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Viscoelastic damping of beams

Mansour Messalti

VISCOELASTIC DAMPING OF BEAMS

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Mansour Messalti

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Approved by:

Dr. Richard B. Hetnarski

Dr. P Marletcar

Dr. J S. Torok

Dr.

DEPARTMENT OF MECHANICAL ENGINEERING

COLLEGE OF ENGINEERING

ROCHESTER INSTITUTE OF TECHNOLOGY

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ABSTRACT

The composite damping characteristics of a viscoelastic sandwich beam are investigated. The sandwich consists of an elastic central layer coated on both sides with a viscoelastic material and two outer layers made out of the same material as the central layer. The composite beam is investigated for damping effectiveness. As in the work of DiTaranto and Kerwin, and also Mead, the energy dissipation due to the vibratory motion of the system is assumed to take place due to the shear deformation of the viscoelastic layer only. The assumption that the outer layers do not stretch or contract during bending, further enhances the composite system's ability to dissipate energy in shear. The assumptions used lead to a fourth order differential equation of motion as opposed to the customary sixth order equation derived in the literature. A closed form solution to the problem of vibration of the composite beam under no load is obtained. A set of boundary conditions are derived using the energy method.

The analytical solution is also shown to reduce to that of an equivalent elastic beam under the same conditions when the viscoelastic term is removed by setting the shear modulus equal to zero. When the same process is used on the composite beam characteristic equation, the result is identical to the frequency equation of an elastic beam.

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LIST OF SYMBOLS

a = Length parameter (dimensionless)

b = Frequency parameter

c = Parameter

d = Imaginary part of the shear modulus

D = Wave number

E = Young´s modulus

G,g = Complex shear modulus

h = Thickness

I = Moment of inertia , Energy integral symbol

i,j,k = Integers

$i^2 = -1$

L = Length

m = Length parameter

M,M = Bending moments

n = Length parameter

p = Complex frequency

r = Frequency parameter

s = Frequency ratio (dimensionless)

S = Shear force

t = Width of beam

u,v = Complex frequency

V = Shear force

W = Displacement

w = Circular frequency

w_o = Natural frequency

x, y = Coordinate axes (Cartesian)

t^* = time value

γ = Shear angle

ρ = Mass per unit length

ϕ_0 = Initial energy

ξ = eigenvalue

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I.1 INTRODUCTION

Surface damping treatments have been used to solve a variety of vibration problems, and in particular when dealing with unwanted resonance noise in sheet metal aircraft structures. These treatments can often be used with today's existing structures be they slated for "high tech" environments such as supersonic jets or "low tech" environments such as a machinery in a manufacturing factory. These surface damping treatments are capable of providing high damping effectiveness over wide ranges of temperature, frequencies, and other parameters that may be affecting the effectiveness of these materials. There are two global types of surface treatments which are classified according to whether the damping material is subjected to extensional or shear deformation.

Surface damping treatments are used in a variety of ways. The most popular way is the sandwich method. This method consists of either having a viscoelastic core with two outside elastic layers, or an elastic core with viscoelastic layers on one or both sides of the elastic layer. The case of alternating layers of elastic and viscoelastic materials has also received much attention in the literature. When an elastic outer layer is used, the method is referred to as the shear type, while in the case of viscoelastic treatment only, the method is referred to as the unconstrained damping method. Nashif [1] reports that for a given weight, the shear type of damping treatment is more efficient than the unconstrained layer damping treatment. He further asserts that the efficiency of the shear type damping treatment is balanced by a greater complication in analysis and application.

The shear type treatment is obtained by adding constraining layers to the unconstrained configuration. These constraining layers may or may not be of similar materials as the central layer. The addition of the constraining layer forces the viscoelastic layer to deform in shear. The shear deformation is the mechanism by which energy is dissipated and greater damping efficiencies are achieved.

This paper investigates the damping effectiveness in the lateral vibration of a viscoelastically layered beam. The sandwich beam is composed of an elastic core, symmetrically coated with a viscoelastic layer, and thin elastic constraining layers. The analysis is based on the fact that there are no loads applied to the composite beam. In this case, it is assumed that the normal force acting on the viscoelastic layer may be neglected, since it is small compared to the normal forces acting on the elastic layers. Also , the damping is due to the shearing of the viscoelastic material, since the modulus of elasticity of the viscoelastic material is several orders of magnitude lower than that of the elastic layers. This is equivalent to assuming that the constraining layers are not stretched or compressed during bending, and that the lengths of the neutral axes of the central and constraining layers are of identical shape during bending. These assumptions lead to the derivation of a fourth order differential equation of motion in lieu of the customary sixth order equation derived in the literature [3]. Upon removal of the above simplifying assumptions, a sixth order equation is obtained.

To facilitate an exact solution to the mathematical model, a suitable set of boundary conditions is derived using the energy method. The differential equation is solved using the Laplace transform method. A closed form solution of the ensuing boundary value problem is obtained. This solution is shown to reduce to that corresponding to an elastic beam upon removal of the viscoelastic term by setting the shear modulus equal to zero. The characteristic equation and its roots are shown to be identical to the elastic case, as given in [2], when the shear parameter in the viscoelastic case is set equal to zero.

I.2 BACKGROUND

As witnessed by the abundance of literature on the subject, vibration damping using viscoelastic materials has received much attention by researchers. Various methods have been applied to characterize the behaviour of these viscoelastically damped systems.

K. Sato and G. Shikanai [3], studied a system consisting of a viscoelastic core and thin elastic outer layers, subjected to axial forces. Their analysis, based on stress and strain equations, yields a sixth order ordinary differential equation of motion set up on the longitudinal strain, instead of one set up on the deflection of the free vibration. They report that the damping efficiency, viewed as a composite loss factor, increases under compression, and decreases with increasing tension. In the absence of an axial force, there is a specific frequency at which the value of the loss factor takes on the maximum value for the sandwich system. The effect of axial forces on the loss factor, appears mainly in the region of frequencies lower than the above mentioned specific frequency, with increasing effects as the frequency decreases. Inversely, this factor decreases gradually as the frequency increases beyond the above mentioned specific frequency.

The fundamental investigation on the problem of vibration damping in sandwich beams under no loads has been performed by Kerwin [4]. In his case, the normal forces acting on the viscoelastic layer were neglected because they were small compared to the normal forces acting on the elastic layer. Also, it was assumed that the damping was primarily due to the shearing of the viscoelastic layer, since the modulus of the elastic layer is several orders of magnitude higher than the modulus of the viscoelastic layer. As a result, the relation for the composite loss factor for an infinitely long or simply supported beam was derived.

Based on the assumptions used by Kerwin, DiTaranto and Blasingame [5,6], have established the relationship between the slope and the axial strain of the elastic layer. This relation was the basis for the analysis used in [3] to determine the composite loss factor and natural frequency of the system. The latter was shown to be independent of the end conditions and mode shape. Further, it was reported by Asnani and Nakra [8] that the damping effectiveness, as described by the equation developed in [3], is dependent on the number of elastic and viscoelastic layers, and the ratios of thicknesses of the viscoelastic layers to the elastic layers. The damping effectiveness of multilayered systems, in an unsymmetrical arrangement of layers, has been studied by Nakra [9].

Ross, Kerwin, and Ungar [10] provided an analysis for a three layer system which is usually used to handle both the extensional and shear damping treatments. This analysis was based on the assumptions that there were rigid connections between layers, and the system was simply supported. Their equations were developed and solved using sinusoidal expansions for the modes of vibration. It was reported that for other boundary conditions, approximations must be used depending on the mode shape af the structure in question [1].

Nashif [11] warns of possible misleading results when dealing with the published values for the complex moduli of viscoelastic materials. He warns that in many cases these moduli have been obtained by using the analysis of Ross, Kerwin and Ungar [10], to estimate the value of the shear modulus instead of using it to estimate the system loss factor, for which it was intended.

II. DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATION OF MOTION

In this paper, a five layer symmetric cantilever beam shown in Fig. 1, is considered. The central elastic layer (1) acts as a main structural element of the beam. The adjoining layers (2) and (2′) are of viscoelastic material with shear modulus $G = G_1(1 + id)$. The outside constraining layers (3) and (3′) are assumed to be thin compared to the central layer ($h_3 \ll h_1$). In addition, it is assumed that the neutral axes of layers (1),(3) and (3′) have identical deflection lines during bending.

In this investigation the lengths of the constraining layers are considered constant at all times during motion, and are equal to the length of the central layer. This is equivalent to the assumption that the constraining layers are not stretched or compressed during bending. In Fig. 2 segment AB lies on the line a_1-a_1, normal to the deflection line of the beam. Line segments AB′ and C′D lie on a line a_2 - a_2 parallel to the y-axis. Obviously, when the beam is not deflected, the lines a_1 - a_1 and a_2 - a_2 coincide. Segments AB′ and C′D lie on the same line if the constraining layer does not stretch and the associated deflections are small.

It follows from $\triangle$ABB′ in Fig. 2 that

$$m \cong y' h_1/2 \tag{1}$$

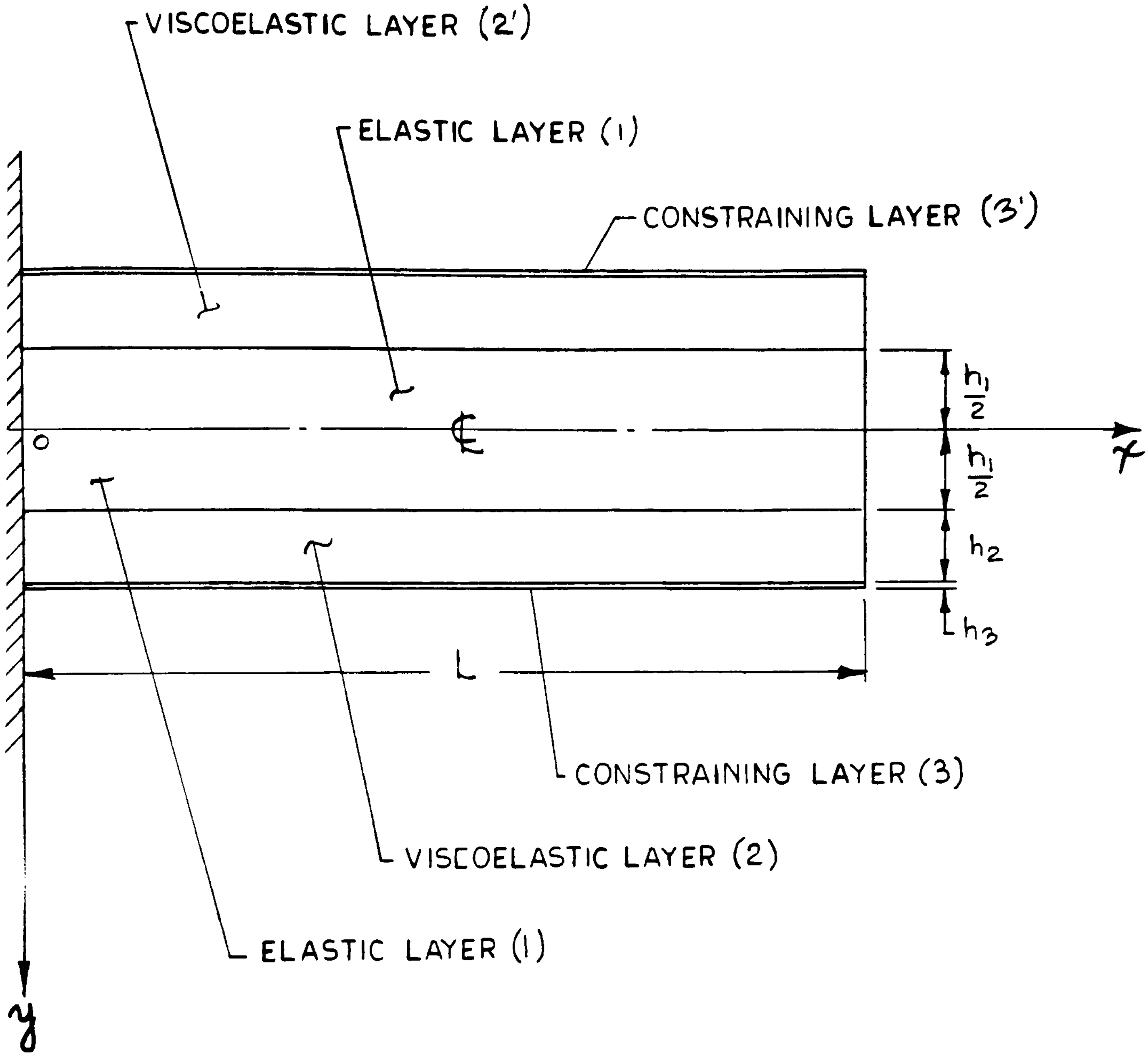


FIG. 1

CANTILEVER BEAM CONSISTING OF FIVE LAYERS

where $y' = \tan \measuredangle BAB' \cong \sin \measuredangle BAB'$ for small deflections.
Similarly, from $\Delta CC'D$, we have

$$n \cong + y'h_3/2 \qquad (2)$$

It is seen, therefore, that the angle of shear γ in the viscoelastic layer is

$$\gamma = y'+(m+n)/h_2 \qquad (3)$$

After substitution ,

$$\gamma = ay' \qquad (4)$$

where

$$a = 1 + (h_1+h_3)/2h_2 \qquad (5)$$

A convention that γ is positive in the counterclockwise direction when measured from the line a_1-a_1 is adopted here. Therefore, the shear force S per unit length of the beam, acting on the surface of layer (1) and, in the opposite direction on the surface of layer (3) is given by

$$S = \gamma G t \qquad (6)$$

where S is in units of force per unit length.

In our case, the bending moment M in an arbitrary cross-section consists of two parts :

$$M = M_0 + M_1 \qquad (7)$$

where M_0 is due to the stiffness of the beam and is given by

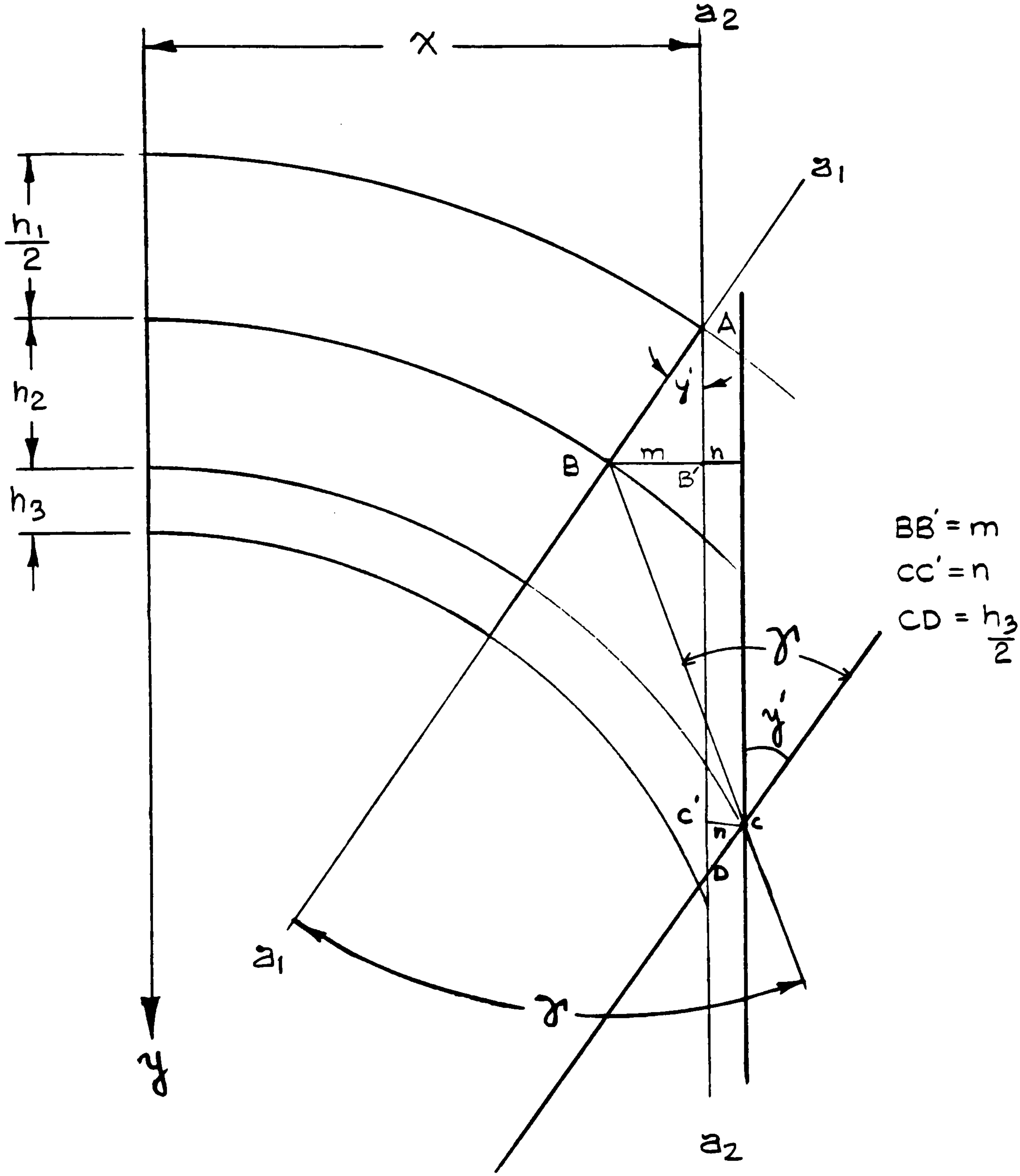


FIG. 2

GEOMETRY OF DISPLACEMENTS

$$M_0 = - EI\, \partial^2 y/ \partial x^2 \tag{8}$$

while M_1 is due to the shear forces S in the viscoelastic layer(see Fig. 3).

In equation (8) the stiffness EI of the layered beam is

$$EI = E_1 I_1 + 2E_3 I_3 \tag{9}$$

Thus, it is assumed that the viscoelastic layers (2) and (2′) do not contribute to the stiffness of the beam.

To establish the equation of motion for the beam, consider Fig. 3 in which the forces and moments acting on an element of length dx of layer (1) are shown. Equating to zero moments about point O, we receive

$$Sh_1 dx - Vdx + dM_0 = 0 \tag{10}$$

From Eq. (10) follows the relation for the shear force V

$$V = dM_0 /dx + Sh_1 \tag{11}$$

On the other hand, from Newton′s second law, the motion of the element in the vertical direction is

$$(\rho dx)\ddot{y} = dV \tag{12}$$

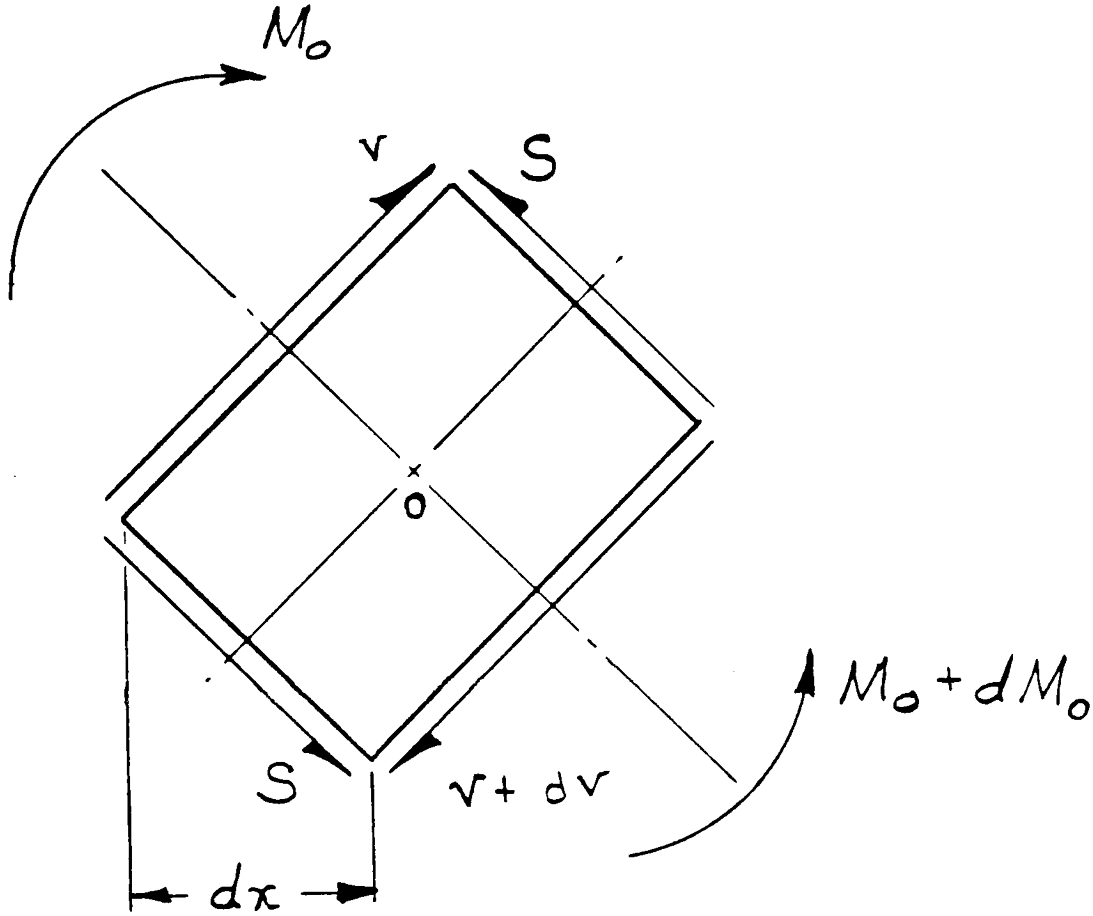


FIG. 3

FREE BODY DIAGRAM

or dividing through by dx we obtain

$$\rho \ddot{y} = V' \tag{13}$$

Differentiating V in (11) and substituting in (13) results in

$$\rho \ddot{y} = M_0'' + h_1 S' \tag{14}$$

Substututing in (14) M from (8) and S from (6) with the expression for γ from (4) we arrive at the equation of motion

$$EIy'''' - ath_1 Gy'' + \rho \ddot{y} = 0 \tag{15}$$

Dividing by the quantity EI we receive

$$y'''' - cG\, y'' + (1/w_0^2)\, \ddot{y} = 0 \tag{16}$$

where

$c = ath_1/EI$

$w_0^2 = EI/\rho$

III ANALYTICAL SOLUTION

III.1 SOLUTION OF THE DIFFERENTIAL EQUATION

To solve the differential equation of motion, the method of separation of variables is first used. Once the variables are separated, the Laplace transform method with respect to x is applied to the ensuing fourth order equation.

The equation governing the vibration of the composite beam is

$$y'''' - cG\, y'' + (1/w_0^2)\, \ddot{y} = 0 \,, \qquad 0 < x < L \tag{17}$$

where

$(w_0)^2 = EI/\rho$ and $c = ath_1/EI$

$G = G_1(1 + id\,)$, $i^2 = -1$

Let $y(x,t) = W(x)\exp(-wt)$, where w is a complex number.

Substitution of $W(x)\exp(-wt)$ into (17) yields

$$W'''' - cG\, W'' + (w/w_0)^2\, W = 0 \tag{18}$$

Apply the Laplace transform to Eq. (18) with respect to the space variable. The transformed function

$$\bar{W}\,(p) = \int_0^{\infty} W(x)\exp(-px)dx$$

satisfies the algebraic equation

$$\{p^4 \bar{W}(p) - p^3 W(0) - p^2 W'(0) - p\, W''(0) - W'''(0)\}$$

$$- cG\, \{p^2 \bar{W}(p) - p\, W(0) - W'(0)\} + (w/w_o)^2\, \bar{W}(p) = 0$$

Solving for $\bar{W}(p)$ we arrive at

$$\bar{W}(p) = (a_1 p^3 + a_2 p^2 + a_3 p + a_4)/(p^4 - rp^2 + s^2)$$

where we have defined

$$r = cG, \ s^2 = (w/w_o)^2, \ a_1 = W(0), \ a_2 = W'(0)$$

$$a_3 = W''(0) - cG\, W(0) = W''(0) - cG\, a_1 \tag{19}$$

$$a_4 = W'''(0) - cG\, W'(0) = W'''(0) - cG\, a_2 \tag{20}$$

Next, express the denominator of $\bar{W}(p)$ as

$$p^4 - rp^2 + s^2 = (p^2 - u^2)(p^2 - v^2) = p^4 - p^2 v^2 - p^2 u^2 + u^2 v^2$$

Equate coefficients of like powers of p and obtain

$$s^2 = u^2 v^2 = (w/w_o)^2, \ r = cG = u^2 + v^2$$

For convenience, denote $(p^2 - v^2)(p^2 - u^2) = D$

The appropriate inverse Laplace transforms are given by

$$L_1 = L^{-1}[p^3/D] = (u^2 \cos ux - v^2 \cos vx)/(u^2 - v^2)$$

$$L_2 = L^{-1}[p^2/D] = (u \sinh ux - v \sinh vx)/(u^2 - v^2)$$

$$L_3 = L^{-1}[p\ /D] = (\cosh ux - \cosh vx)/(u^2 - v^2)$$

$$L_4 = L^{-1}[1/D] = (v \sinh ux - u \sinh vx)/\{uv(u^2 - v^2)\}$$

Therefore

$$W(x) = L^{-1}\,\bar{W}[p] = a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 L_4 \tag{21}$$

The boundary conditions shown below are derived in Appendix A using the energy method.

$$W = 0 \quad \text{at} \quad x = 0 \tag{22}$$

$$W' = 0 \quad \text{at} \quad x = 0 \tag{23}$$

$$W''' - cG\, W' = 0 \quad \text{at } x = L \tag{24}$$

$$W'' = 0 \quad \text{at } x = L \tag{25}$$

III.2 SOLUTION OF THE ASSOCIATED BOUNDARY VALUE PROBLEM

Upon insertion of the boundary conditions at $x = 0$ into Eq. (21) we get

$$a_1 = a_2 = 0$$

and thus

$$W(x) = a_3 L_3 + a_4 L_4$$
$$= [1/(u^2 - v^2)]\ [a_3(\cosh ux - \cosh vx) + (a_4/uv)(v \sinh ux - u \sinh vx)]$$

The respective derivatives of the solution are

$$W'(x) = [1/(u^2 - v^2)][a_3(u \sinh ux - v \sinh vx) + a_4(\cosh ux - \cosh vx)]$$

$$W''(x) = [1/(u^2 - v^2)][a_3(u^2 \cosh ux - v^2 \cosh vx) + a_4(u \sinh ux - v \sinh vx)]$$

$$W'''(x) = [1/(u^2 - v^2)][a_3(u^3 \sinh ux - v^3 \sinh vx) + a_4(u^2 \cosh ux - v^2 \cosh vx)]$$

From Eq. (23), it follows that

$$a_3 = W''(0) - cG\ W(0) = W''(0)$$

From Eq. (24), it follows that

$$a_4 = W'''(0) - cG\ W'(0) = W'''(0)$$

Using $W''(L) = 0$, we obtain

$$[1/(u^2 - v^2)][W''(0)(u^2 \cosh uL - v^2 \cosh vL) + W'''(0)$$
$$(u \sinh uL - v \sinh vL)] = 0 \qquad (26)$$

The fourth boundary condition $W'''(L) - cG\, W'(L) = 0$ leads to

$$[W''(0)/(u^2 - v^2)][(u^3 \sinh uL - v^3 \sinh vL)] + [W'''(0)/(u^2 - v^2)] \times$$

$$(u^2 \cosh uL - v^2 \cosh vL) - [cG\, W''(0)/(u^2 - v^2)][u \sinh uL - v \sinh vL)]$$

$$- [cG\, W'''(0)/(u^2 - v^2)][\cosh uL - \cosh vL] = 0 \qquad (27)$$

Equations (26) and (27) constitute a set of simultaneous (homogeneous) equations in $W''(0)$ and $W'''(0)$. The determinant of the coefficients results in the following transcendental equation for the determination of u and v :

$$(u^4 + v^4)(\cosh uL \cosh vL) - uv(u^2 + v^2)(\sinh uL \sinh vL) = 2\, u^2 v^2 \qquad (28)$$

Equation (28) is the frequency equation for the damped composite beam. Equation (28) shows two unknowns u and v. The second equation relating the two unknowns u and v is $u^2 = c\, G_1(1 + id\,) - v^2$.

Insertion of the above value of u^2 into Eq. (28) leads to one equation with one unknown with complex coefficients and complex arguments for the hyperbolic functions.

From Eq. (26), it follows that

$$W'''(0) = - W''(0)(u^2 \cosh uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL) \tag{29}$$

Equations (19), (20) and (29) inserted into Eq. (21) lead to

$$W(x) = [W''(0)/(u^2 - v^2)][\cosh ux - \text{cohs}\ vx - A\ (v \sinh ux - u \sinh vx)/uv] \tag{30}$$

where

$$A = (u^2 \text{coh}\ uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL)$$

Equation (30) represents the solution of the boundary value problem. For a proof refer to Appendix C.

The values of u and v for an elastic beam, as determined in Appendix D, are given by

$$u_1 = b, \quad u_2 = - b$$

$$v_1 = ib, \quad v_2 = - ib$$

where

$$b = (is)^{1/2}, \quad s^2 = (w/w_0)^2 \text{ and } \quad (i)^2 = -1$$

IV. COMPARISON WITH AN EQUIVALENT ELASTIC BEAM

The object of this comparison is to show that if the above assumptions are correct, the purely elastic case (Eqs. (31) through (35)) would be obtained upon removal of the viscoelastic term by setting the shear parameter cG equal to zero in Eqs. (17) and (22) through (25).

The boundary value problem for a cantilever beam as given in [2] is stated as

$$W'''' - b^4 W = 0, \quad W = W(x), \qquad 0 < x < L \tag{31}$$

$$W = 0 \qquad \text{at} \quad x = 0 \tag{32}$$

$$W' = 0 \qquad \text{at } x = 0$$

$$W'' = 0 \qquad \text{at } x = L \tag{33}$$

$$W''' = 0 \qquad \text{at } x = L$$

The characteristic equation is given by

$$\cosh bL \cos bL = -1 \tag{34}$$

Here again $b^4 = (w/w_o)^2$.

Comparison of Eqs. (17) and (31) shows that the two equations differ only by the term which is introduced by the viscoelastic damping in the case of Eq. (17) (i.e. cGW'').

It is easily verified that when $cG = 0$ in Eqs. (17) and (22) through (25), the two boundary value problems are identical. Next, we need to compare their characteristic equations (28) and (34), respectively.

The characteristic equation of the compound beam as derived above is given by Eq. (28):

$$(u^4 + v^4)(\cosh uL \cosh vL) - uv(u^2 + v^2)(\sinh uL \sinh vL) = 2\,u^2 v^2$$

Recall that

$$cG = (u^2 + v^2)$$

Now let us square both sides of the equation above to obtain

$$(cG)^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2\,u^2 v^2, \text{ or } (cG)^2 - 2\,u^2 v^2 = u^4 + v^4$$

Substitution of cG and $(cG)^2 - 2u^2 v^2$ into Eq.(28) yields

$$[(cG)^2 - 2u^2 v^2)] \cosh uL \cosh vL - uv(cG)(\sinh uL \sinh vL) = 2\,u^2 v^2$$

The next step is to remove the shear term by setting $cG = 0$ in the above equation and obtain

$$-2\ u^2 v^2 (\cosh uL\ \ \cosh vL) = 2\ u^2 v^2$$

Divide the above equation by $-2\ u^2 v^2$ and obtain

$$\cosh uL \cosh vL = -1$$

Upon insertion of the roots ($u_1 = \pm\ b$, $v_1 = \pm\ ib$) into the above equation, and noting that $\cosh(\pm ibL) = \cos(bL)$, the characteristic equation of an elasic beam [2] is obtained, namely

$$\cosh bL \cos bL = -1$$

The next step is to check if the solution derived for the multilayer case reduces to that of a purely elastic case upon setting cG equal to zero when determining the complex roots u and v and inserting those values in Eqs. (30). The purely elastic case is given by Eqs. (31) through (34).

The above procedure was carried out and the solution was found to be unique for all possible combinations of the roots (see appendix E).

$$W(x) = [W''(0)/2b^2 (\sin bL + \sinh bL)][(\cosh bx - \cos bx) \times (\sinh bL + \sin bL) - (\cosh bL + \cos bL)(\sinh bx - \sin bx)].$$

This solution is what equation (30) reduces to upon setting cG equal to zero and is the solution to the purely elastic case.

The final solution to Eq. (17) with its associated boundary conditions (22) through (25) is given by

$$y(x,t) = W(x) \exp(-wt)$$

$$= [W''(0)/(u^2 - v^2)][\cosh ux - \cosh vx - A (v \sinh ux - u \sinh vx)] \exp(-wt)$$

where

$$A = (u^2 \cosh uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL)$$

and

w is the damped complex frequency obtained by solving equation (28).

V. COMPUTATIONAL EXAMPLE

In order to obtain plots of the response y(x,t), roots of the transcendental equation (28) have to be computed first. Once the roots are obtained, they will be inserted into equation (30) for the purpose of generating plots.

We begin by evaluating the complex frequencies u and v. To this effect we will convert the equations of the frequencies from their cartesian forms to their equivalent polar forms. This conversion enables us to handle the imaginary part under the radical.

The value of u and v is (Appendix D)

$$u = \pm\{(1/2)[r + (r^2 - 4s^2)^{1/2}]\}^{1/2}$$

$$v = \pm\{(1/2)[r - (r^2 - 4s^2)^{1/2}]\}^{1/2}$$

Noting that $(-4s^2)$ can be expressed as $(i2s)^2$, the above equations are presented as

$$u = \pm\{(1/2)[r + (r^2 + (i2s)^2)^{1/2}]\}^{1/2}$$

$$v = \pm\{(1/2)[r - (r^2 + (i2s)^2)^{1/2}]\}^{1/2}$$

Observe that $[r^2 + (i2s)^2]^{1/2}$ as shown in Fig. 5 is equal to $r/\cos\theta$ where

$$\tan\theta = (i2s/r) = [i2w/(w_0 cG)]$$

$$u = \pm [(1/2)(r + r/\cos\theta)]^{1/2} = \pm [r/2(1 + 1/\cos\theta)]^{1/2}$$

Similarly

$$v = \pm (r/2)^{1/2} \cos\alpha$$

where

$$r = cG$$

and

$$\tan\alpha = i/(\cos\theta)^{1/2}$$

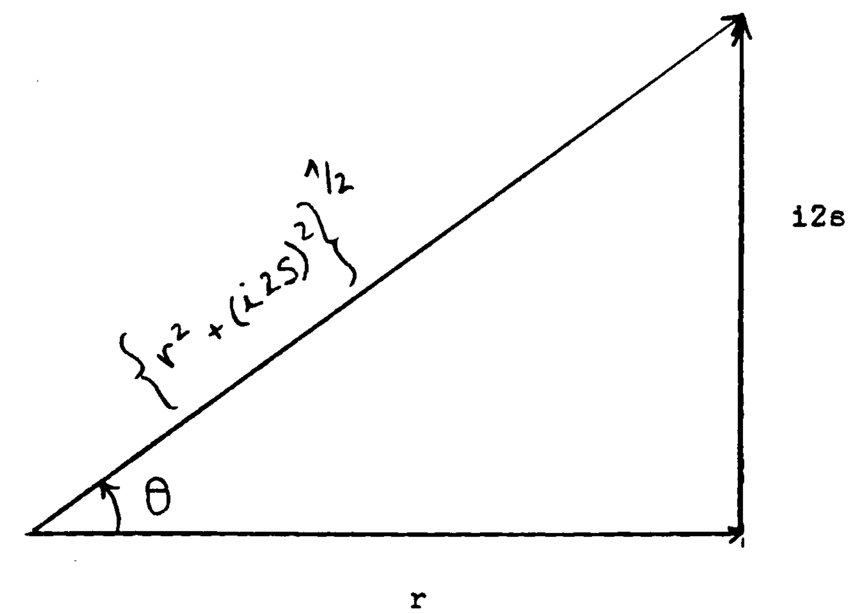


Fig.5. Polar representation of the frequencies u.

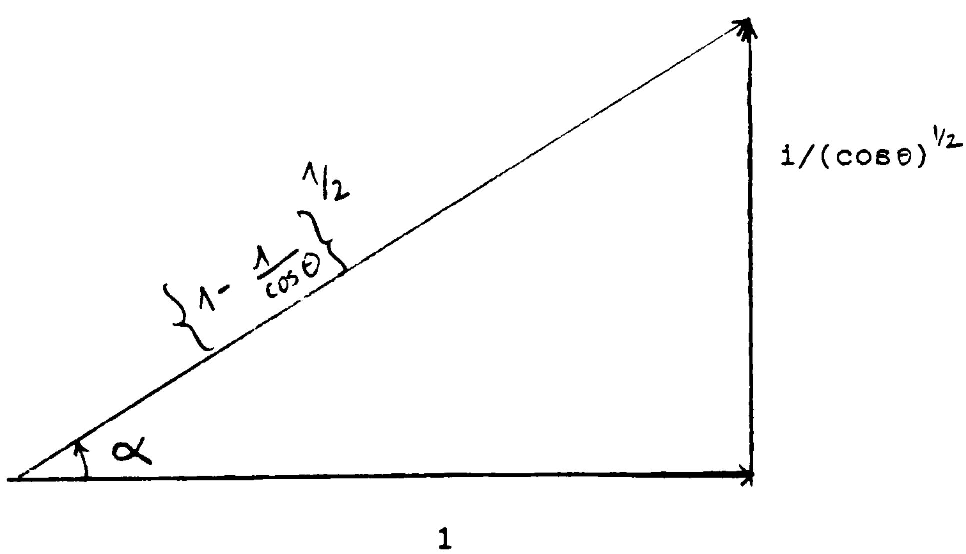


Fig.6.Polar representation of the frequencies v.

Numerical example

In order to use the equations developed for the complex frequency parameters u and v , the flexural rigidity and the natural frequency of the system, the shear modulus of the viscoelastic material, and the width parameter "c" will be evaluated.

1. Evaluation of the shear modulus G.

The equation relating the shear modulus G to frequency for the viscoelastic material SYLGARD 188 manufactured by Dow Corning, is determined from data sheets 006 and 007, shown in [1], page 380.

$$G_1 = 800(w/20)^{.234}$$

$$d = 0.55$$

$$G = G_1(1 + id) = 800(w/20)^{.234}[1 + i\ (0.55)] \qquad (F1)$$

2. Evaluation of the stiffness of the composite beam, EI.

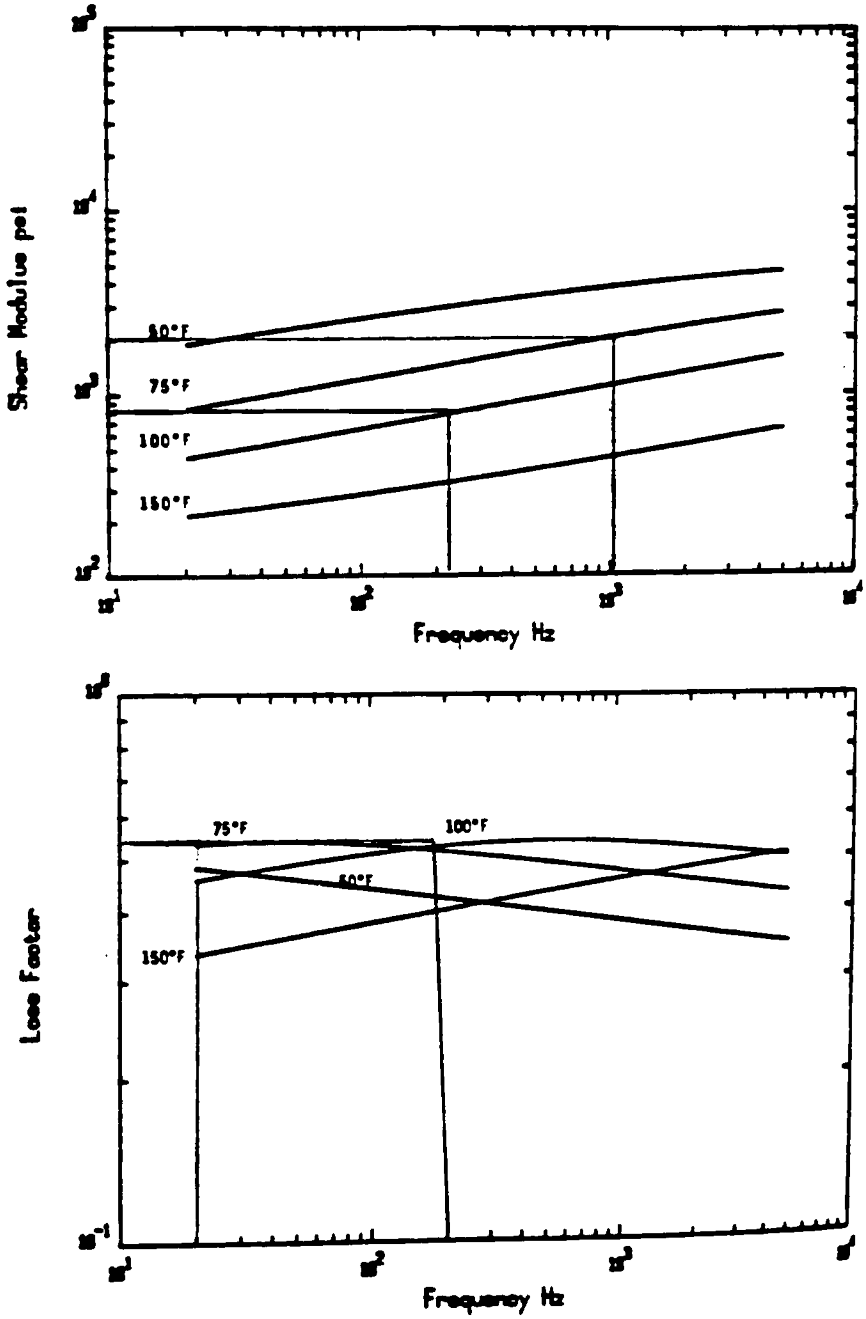
$$EI = E_1 I_1 + 2\ E_3 I_3 \qquad (9)$$

$$E_1 = E_3 = 30 \times 10^6 \quad \text{(psi)}$$

$$I_1 = t(h_1^3)/12\ ; \quad I_3 = t(h_3^3)/12.$$

$$EI = 2.5 \times 10^6\, t\ [(h_1^3) + 2(h_3^3)] \qquad (F2)$$

Data Sheet 006. Damping properties of Dow Corning Sylgard 188 (continued).



006B. Damping properties with frequency.

Fig. 7 DAMPING PROPERTIES OF SYLGARD [2] PP 380

3. Evaluation of the parameter c.

Recall that the parameter c is defined as $c = ath_1/EI$

$$a = 1 + (h_1+h_3)/(2h_2) \tag{5}$$

Insert (F2) and (5) into c and receive

$$c = ath_1/EI = 12\ ath_1/E(steel)t[(h_1^3) + 2(h_3^3)]$$

or

$$c = (2h_1h_2 + h_1^2 + h_1h_3)/(2.5 \times 10^6)(2h_2)(h_1^3 + 2h_3^3) \tag{F3}$$

Let the central layer thickness $h_1 = 0.10''$

$$c = (0.01 + 0.2h_2 + 0.1h_3)/[5 \times 10^6 (0.001 + 2h_3^3)\ h_2] \tag{F4}$$

4. Evaluation of the natural frequency w_o.

$(w_o)^2 = EI/\rho$ where ρ is the mass per unit length and is equal to

$$\rho = 0.000731\ t\ (h_1 + 2h_3)\quad \text{(slugs/inch)}$$

Insert eq. (F3) into w_o and receive

$$w_o = \{[30 \times 10^6 (t/12)(h_1^3 + 2h_3^3)]/[0.000731\ t\ (h_1 + 2h_3)]\}^{1/2}$$

Note that the beam width appears in both the numerator and the denominator, therefore it does not play any role at all.

Again let us assume the central layer thickness $h_1 = 0.100''$ and wo reduces to

$$w_o = 58497.75\ [(0.001 + 2h_3^3)/(0.10 + 2h_3)]^{1/2} \tag{F5}$$

Case study

Let $h_1 = 0.10''$, $h_2 = 0.10''$ and $h_3 = 0.01''$. Observe that $h_3 << h_1$. Substitution of the above values into equations (F5) and (F4) yields the values of w and the parameter "c" respectively.

$$w = 5345.42 \quad \text{rad/sec}$$

$$c = 42.112 \times 10^{-6} \quad (1/\text{lb})$$

Examination of the complex frequencies u and v , reveals that they differ only in the sign of the second radical. Should this radical become null, the complex frequencies would take on the following value:

$$u = \pm (cG/2)^{1/2}$$

and

$$v = \pm (cG/2)^{1/2}$$

Recall that the response of the system has the quantity $(u^2 - v^2)$ appearing in the denominator, and the quantity (cosh uL - cosh vL) appearing in the numerator. Choosing u = v or u = - v, leads to the indeterminate form of 0/0. The value of w leading to the above situation is 139.548 rad/sec. This value is only valid for the case study shown above. This method is straightforward and simple. First determine the unwanted frequencies, then through trial and error determine the various layers´ thicknesses that most efficiently dissipate the vibratory energy.

VI. TYPICAL OUTPUT

```
omega:    1.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.33091462626920E-002,   1.11628668253210E-002)

omega:    2.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.45691042796281E-002,   1.21628360049493E-002)

omega:    3.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.54544514154905E-002,   1.28277640451234E-002)

omega:    4.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.61906149494153E-002,   1.33526626802755E-002)

omega:    5.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.68471709240383E-002,   1.37997543376747E-002)

omega:    6.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.74537639490607E-002,   1.41966702720462E-002)

omega:    7.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.80253537597980E-002,   1.45580063960702E-002)

omega:    8.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.85703920877658E-002,   1.48923941794902E-002)

omega:    9.00000000000000E+000
h2:    5.00000000000000E+000
omegaO:    5.58032969196881E+004
(u,v) = (   1.90940832488420E-002,   1.52053735316618E-002)
```

```
omega:    1.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.55973746487816E-002,   1.30979487895781E-002)

omega:    2.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.70008461606384E-002,   1.42406753347639E-002)

omega:    3.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.79415477267452E-002,   1.49806621543263E-002)

omega:    4.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.86929110770249E-002,   1.55507581475895E-002)

omega:    5.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.93424117062578E-002,   1.60265508973355E-002)

omega:    6.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   1.99288052675524E-002,   1.64421827049771E-002)

omega:    7.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   2.04723473652626E-002,   1.68159233769211E-002)

omega:    8.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   2.09847730187148E-002,   1.71586460466372E-002)

omega:    9.00000000000000E+000
h2:    1.00000000000000E+000
omega0:    5.58032969196881E+004
(u,v) = (   2.14733899745654E-002,   1.74773218754508E-002)
```

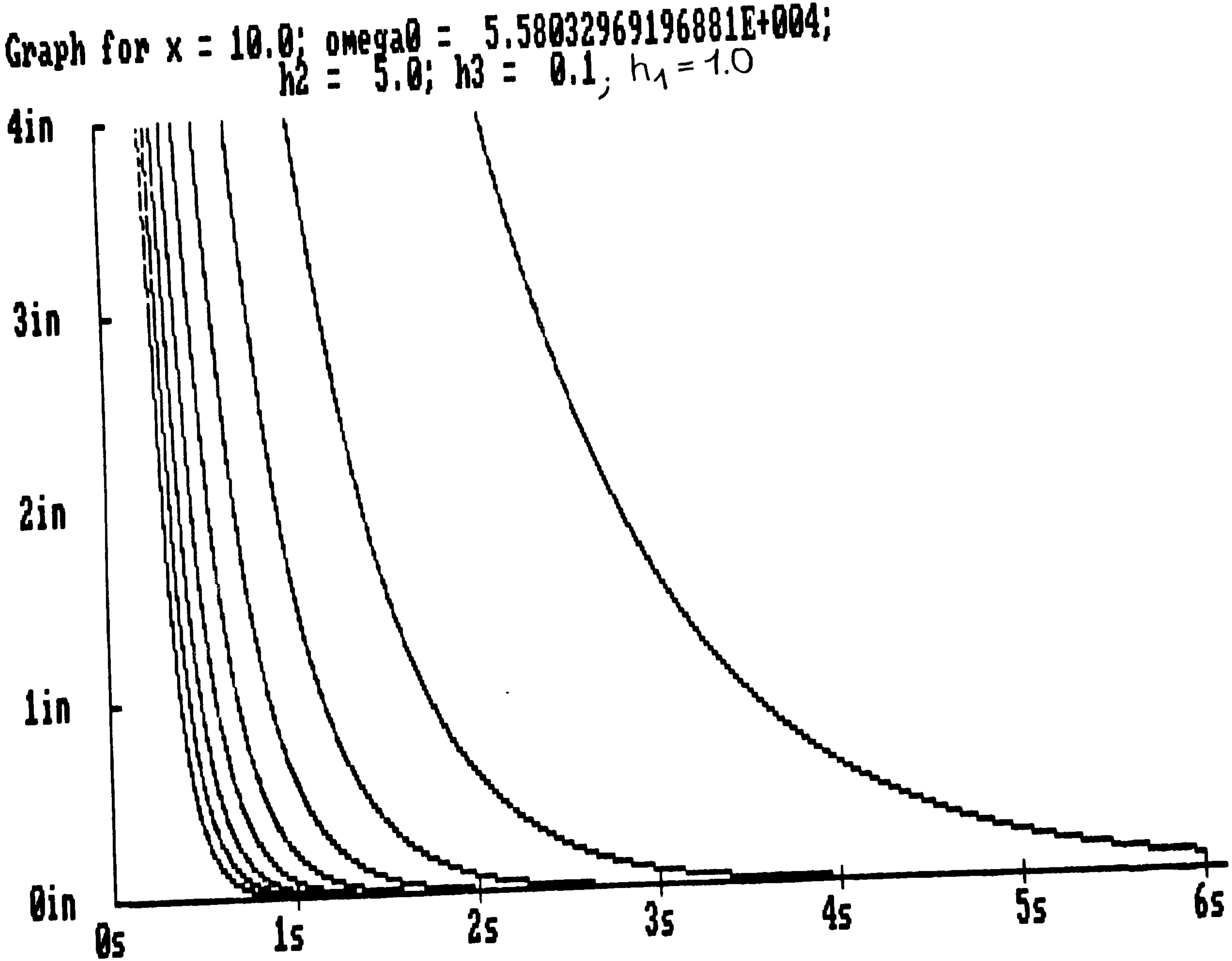


Fig. 8 PLOT OF y(x,t) VS. TIME

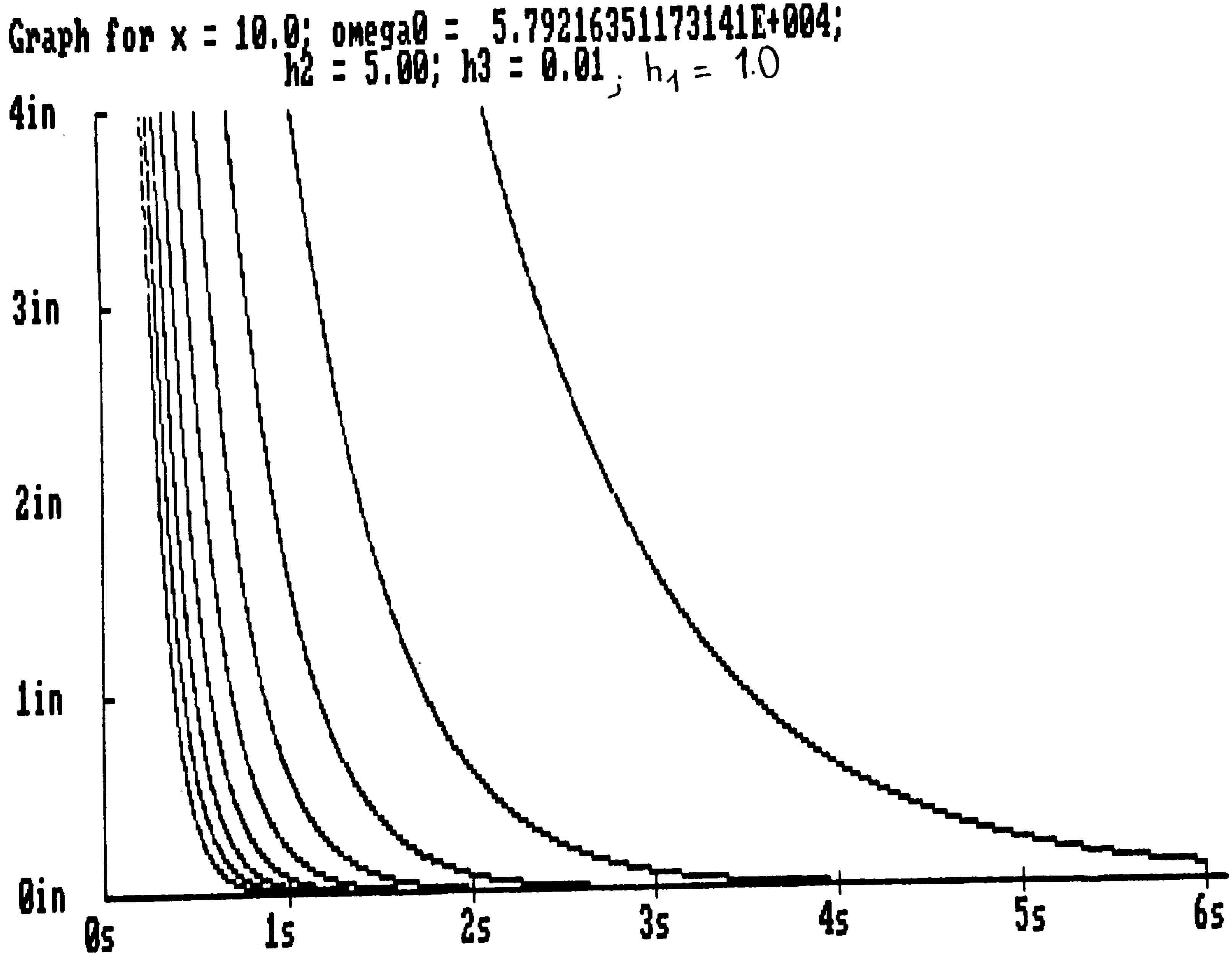


Fig. 9 PLOT OF y(x,t) VS. TIME

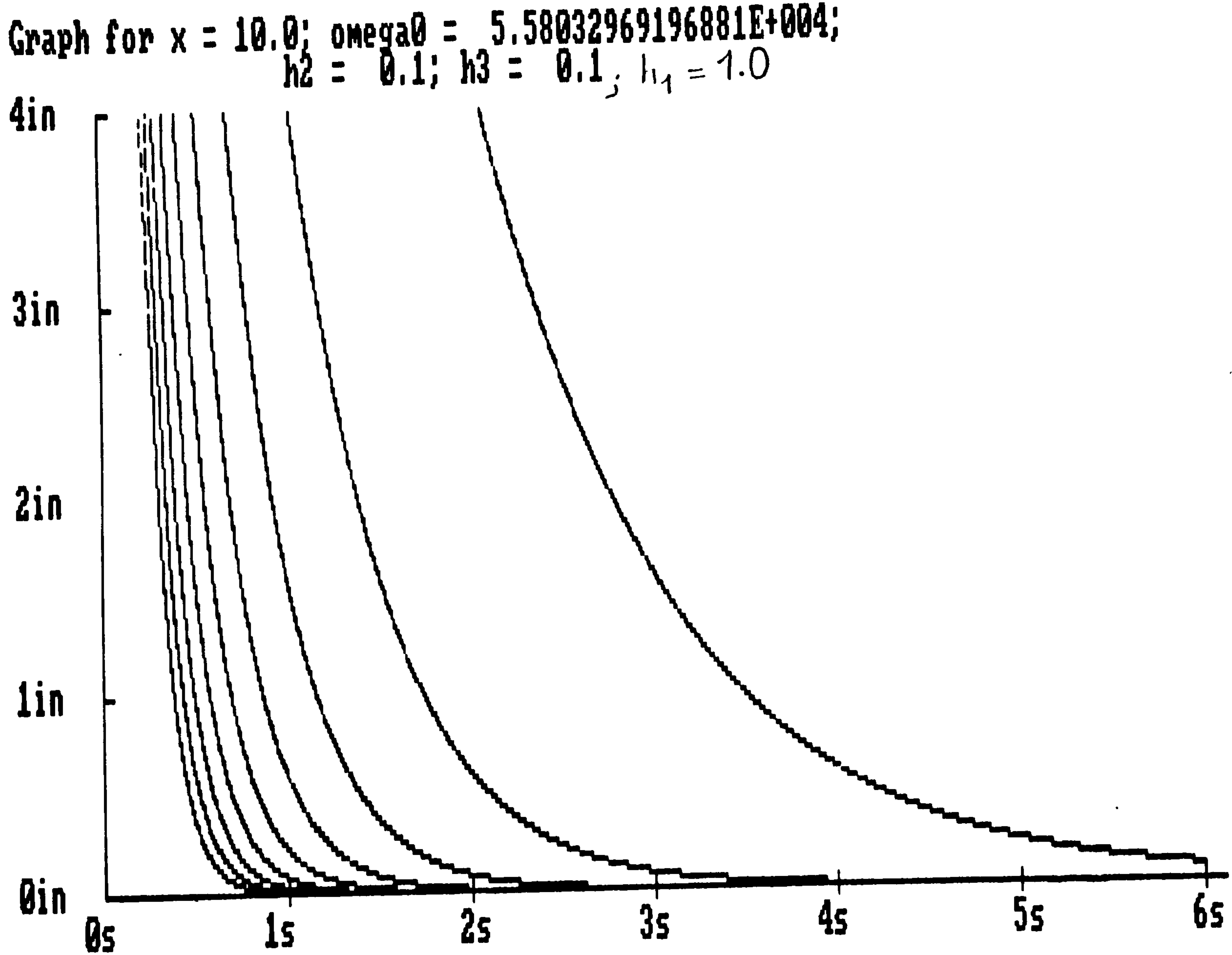


Fig. 10 PLOT OF y(x,t) VS. TIME

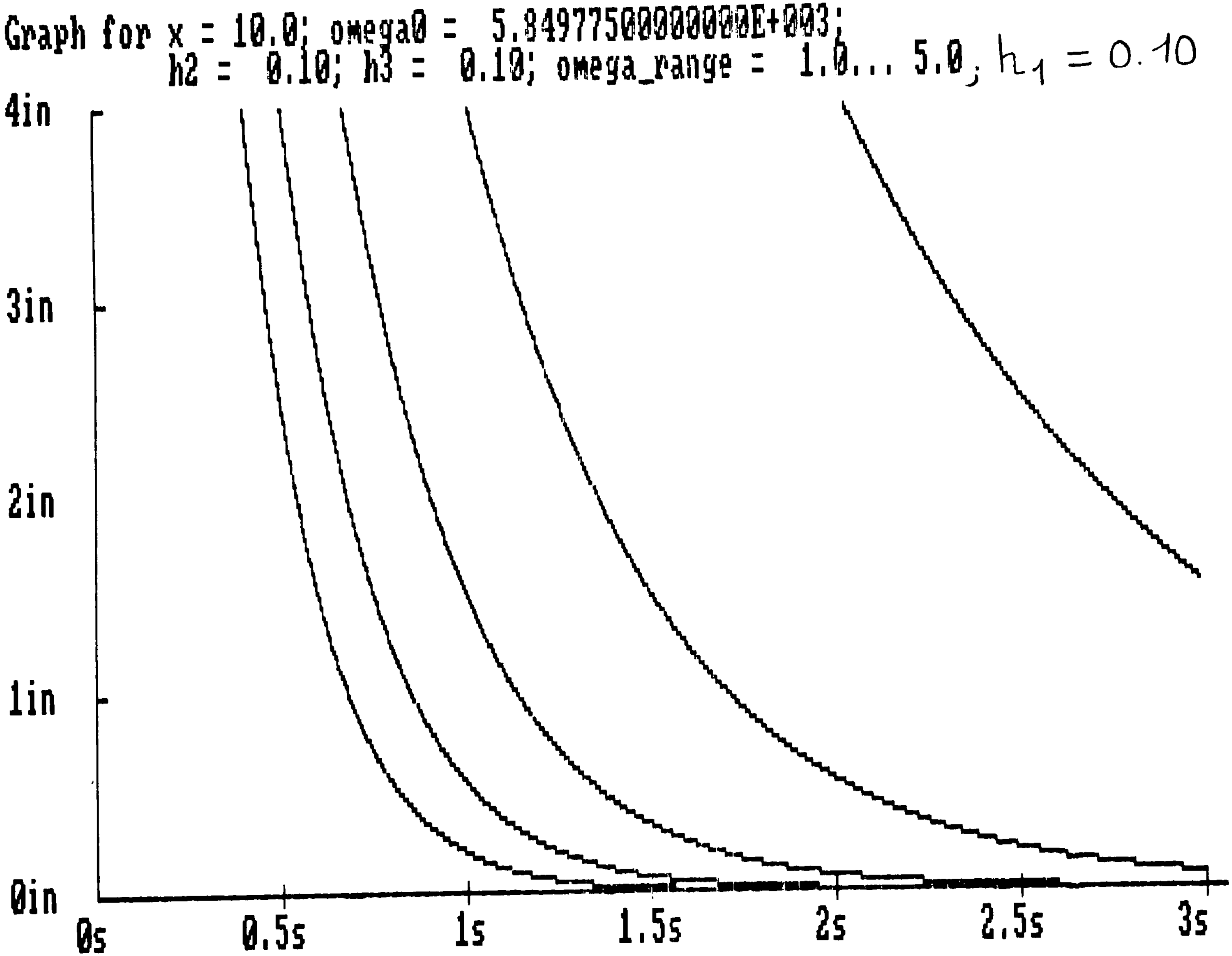


Fig. 11 PLOT OF y(x,t) VS. TIME

VII. DISCUSSION

When assessing the effect of the different layers' thicknesses on the system response $y(x,t)$, the following points can be made:

One point is from the computational example, where we can see that in order to take advantage of the shear modulus we would have to use high frequencies, but due to the computational difficulties posed by the evaluation of the complex frequencies u and v, we have to limit ourselves to $w < 139.55$ rad/sec.

Another point worth mentioning is that higher values of w, which allow a greater benefit from the shear modulus, also make the exponential decay term $\exp(-wt)$ approach the value zero very quickly. Also, it is noticed that u and v are so close to each other that the value of $1/(u^2-v^2)$ in Eq. (30) make the values of the function $y(x,t)$ very large.

Due to the above difficulties, the imaginary part of the shear modulus, being so small in comparison to its real part, is ignored. Without loss of generality, the results indicate that there is a slight variation in the response $y(x,t)$ for various thicknesses of the viscoelastic layer.

Further analysis shows that upon setting the viscoelastic term (cG) equal to zero in equations (28) and (30), the response of the equivalent elastic beam and its characteristic equation are deduced. The analysis also shows that the complex frequencies are weak functions of the constraining and viscoelastic layers which is evident when trying to assess the effect of these layers' thicknesses on the system response y(x,t).

To further improve on this analysis, we would have to pay attention to the following:

Recall that

$$u = \pm \{(1/2)\,[cG + [(cG)^2 - (2w/w_o)^2]^{1/2}\}^{1/2}$$

and

$$c = ath_1/EI; \quad a = 1 + (h_1 + h_3)/2h_2 \ .$$

If one desires to revert back to the elastic case, h_2 or G or both would have to be set to zero to remove the viscoelastic term. the removal of the viscoelastic term in the differential equation (17), produces a differential equation which is identical to that of the purely elastic case. The removal of the shear modulus G from the multilayer case characteristic equation (28), produces also a characteristic equation corresponding to that of an elastic beam. Now it is obvious that the removal of the damping can also be achieved by setting the thickness of the viscoelastic layer equal to zero. By inspection of the equations for the evaluation of the complex frequencies $u_{1,2}$, $v_{1,2}$, we conclude that the parameter "a" approaches infinity as h_2 approaches zero. This fact, in turn, makes the parameter c approach infinity which renders the response identically zero instead of producing the expected response of the elastic case.

This may suggest that if the system is purely elastic, then it cannot be overdamped, and the reverse is also true. There is no passage from an overdamped case to an undamped one. This result is observed when trying to obtain the response of a purely elastic beam by setting the viscoelastic thickness equal to zero in the multilayer system. When starting with the equations of the multilayer system, setting the thicknesses of the viscoelastic layers equal to zero generates no plots. This fact suggests that the passage from an overdamped system to a purely elastic system is not possible in this case.

Another point of interest is the difference between the value of the composite beam flexural rigidity EI used in this analysis, and the one commonly used in the literature [1]. The flexural rigidity of a three layer system of Fig. 4 as given by Ross, Kerwin and Ungar (R.K.U.) [1] has the form

$$EI = (1/12)(E_1 h_1^3 + E_2 h_2^3 + E_3 h_3^3) - (E_2 h_2^2/12)(h_{31} - D)/(1+g) + E_1 h_1 D^2$$

$$+E_2 h_2 (h_{21} - D)^2 + E_3 h_3 (h_{31} - D)^2 - [(1/2)E_2 h_2 (h_{21} - D)$$

$$+E_3 h_3 (h_{31} - D)](h_{31} - D)/(1+g)$$

where

$$h_{31} = (1/2)(h_1 + h_3) + h_2$$

$$h_{21} = (1/2)(h_1 + h_2)$$

$$g = G_2/E_3 h_3 h_2 p^2$$

$$D = [E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})]/[E_1 h_1 + E_2 h_2/2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)]$$

$p = (\xi n/L)$ and is called the wave number.

D = distance from the neutral axis of the three layer system to that of the original beam, h_1.

Comparing the above flexural rigidity with the one used in this analysis we can easily see the difference. Note that to account for the other two layers in our case, we would have to double all elements of EI in the R.K.U. analysis except the part pertaining to the central layer. When dealing with this configuration, one generally uses a complex Young's modulus E while we elected to use only its real part.

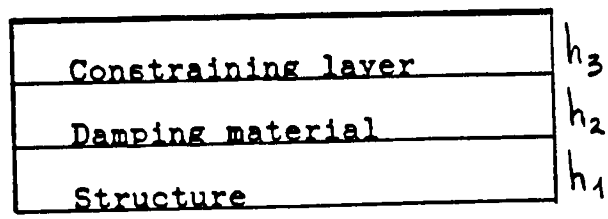


Fig. 12 Constrained layer damping used in the R.K.U. analysis

VIII. LOSS FACTOR AND ITS RELATION TO DAMPING

The loss factor η is related to the vibration frequency of the system by

$$w^2 = w_1^2(1 + i\eta) \qquad (35)$$

where

w = Damped frequency of the system

and

η = Loss factor of the system.

To derive the expressions for the frequency of the system, and subsequently the equations for the loss factor and damped frequency equation for the composite beam, the denominator of the transformed equation is set equal to zero to allow for a partial fraction decomposition of the transformed solution (Appendix D).

Thus, we require $p_i^4 - cG\, p_i^2 + (w/w_0)^2 = 0$

Solve for w^2 and obtain:

$$w^2 = w_0^2(cG\, p_i^2 - p_i^4)$$

Recall that $G = G_1(1 + id)$

Substitute into the above equation and obtain

$$w^2 = w_0^2(cG_1 p_i^2 - p_i^4 + i\, cG_1 p_i^2 d)$$

The above equation rearranged yields

$$w^2 = w_o^2 p_i^2 (cG_1 - p_i^2)[1 + i\, cG_1 d/(cG_1 - p_i^2)] \tag{36}$$

Equate the real and imaginary parts of Eq. (35) and (36) and obtain

$$w_1^2 = w_o^2 p_i^2 (cG_1 - p_i^2) \tag{37}$$

and

$$\eta = cG_1 d/(cG_1 - p_i^2) = d/(1 - p_i^2/cG_1) \tag{38}$$

Note that as defined, the loss factor has to be a real positive number. If the loss factor is negative, this means that the vibrations are being amplified instead of being damped. The condition put on the loss factor requires that $cG_1 > p_i^2$. If p_i^2/cG_1 is very small, then the loss factor of the composite system is approximately equal to the loss factor of the viscoelastic material.

Next we will express the damped frequency and the loss factor of the composite system as a function of the ratios of the various layers of the composite beam.

Recall that

$$c = (2h_1 h_2 + h_1^2 + h_1 h_3)/[kh_2(h_1^3 + 2h_3^3)] \tag{39}$$

where $k = 5 \times 10^6$

Define $H = h_1/h_2$; $H_1 = h_1/h_3$; $H_2 = h_2/h_3$

Divide both the numerator and denominator of Eq. (39) by $h_2 h_3^3$ and use the above definitions for the thickness ratios to obtain

$$c = (1/kh_3^2)[2H_1 + HH_1 + H]/[H_1^3 + 2] \tag{40}$$

Using the same process,

$$w_o^2 = EI/\rho = (E/\rho)\,(h_3^2/12)\,(H_1^3 + 2)/(H_1 + 2) = k\, h_3^2 (H_1^3 + 2)/(H_1 + 2) \qquad (41)$$

where $k = E/12\rho = 30 \times 10^6 / 12 \times 0.000731$

Insertion of Eqs. (40) and (41) into Eqs. (37) and (38) yields the desired result, namely

$$w_1^2 = (G_1 p_i^2 k_1/k)(2H_1 + HH_1 + H)/(H_1+2) - p_i^4 k_1 h_3^2 (H_1^3 + 2)/(H_1+2) \qquad (42)$$

$$\eta = d/\{1 - p_i^2/[k\, h_3^2 (2H_1 + HH_1 + H)/(H_1^3 + 2)]\} \qquad (43)$$

Equations (42) and (43) are the damped frequency squared and loss factor of the composite system respectively. They are expressed as functions of the thickness ratios of the various layers.

Equations (42) and (43) are also functions of the roots of the characteristic equation, namely p . The values of p_i will be determined from the transcendental equation (28). Upon inspection of equation (28), we observe that we have two unknowns. The second equation which will be used is the relation between the two unknowns and the viscoelastic term cG. The shear modulus is a complex number, therefore Eq. (28) is also complex.

In the following paragraph, we will outline a possible method of solving for the roots of the transcendental equation.

Recall that

$$p_{1,2} = u_{1,2}, \quad p_{3,4} = v_{1,2}$$

Recall also Eq. (28)

$$(u^4 + v^4)\cosh uL \cosh vL - uv(u^2 + v^2)\sinh uL \sinh vL = 2 u^2 v^2 \qquad (28)$$

and

$$u^2 + v^2 = cG_1(1 + id)$$

Express $u^2 = cG_1(1 + id) - v^2$

Insert the above value of u into Eq. (28) and arrive at the desired result. The solution of the above equation may be achieved by separating the real and imaginary parts and solving the two resulting equations numerically. The work required is beyond the scope of this paper.

IX. CONCLUSIONS

For the case of a five layer system, the governing differential equation is derived. Its associated boundary conditions are obtained using the energy method. The technique of solving the problem is novel in that the Laplace transform method is used in a finite space domain.

The solution of the ensuing boundary value problem, its characteristic equation, and the complex frequencies are shown to reduce to that of an elastic system when the viscoelastic term cG is set to zero.

We found that passage from an overdamped system to a purely elastic system is not possible. This is observed when the viscoelastic layers' thicknesses are set to zero in the equations related to the multilayer system.

Since w is assumed to be complex, the equations derived analytically are of a general nature. When an attempt is made to generate plots, it becomes evident that w has to be real. Taking w real causes the system to become overdamped.

To assess the effect of thickness ratios of the various layers on the system response, the viscoelastic and constraining layers' thicknesses were varied while the central layer thickness was held constant. The plots generated show no apparent change. A possible reason may be that the complex frequencies do not change rapidly as w is varied from 1 to 10 rad/sec. We have arbitrarily set $W''(0) = 1$ in the system response, which means that the moment at $x = 0$ is not zero. $W''(0)$ could have been set to any constant other than zero.

Using the second program, the plots generated appear to be similar to the ones obtained with the first program, and show that the system is more efficient in the higher frequency range than in the lower frequency range.

The importance of an overdamped system lies in its ability to protect highly sensitive machines such as laser printers and computer disk drives. It is also of importance in a host of other applications where mass is critical, such as aircraft vibration damping and isolation of navigation systems. Further study may reveal that with the advance of composite materials, viscoelastic polymers, and building supplies, the use of multilayered beams in structures will be useful in damping the shock waves resulting from earthquakes.

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APPENDIX A

DERIVATION OF THE BOUNDARY CONDITIONS

The boundary conditions are determined using the energy approach.

Utilizing the left hand side of (18), compute the integral

$$I = \int_0^L [W_i'''' - cG\, W_i'' + (w/w_o)^2 W_i]\, W_j\, dx \qquad \text{(A1)}$$

Where W_i and W_j are functions of x only, and subscripts i and j denote node numbers. Integrating (A1) repeatedly by parts, the following result is obtained

$$I = \int_0^L [W_j'''' - cG\, W_j'' + (w/w_o)^2 W_j]\, W_i\, dx$$

$$+ [W\ '''W - W\ ''W\ ' + W\ 'W\ '' - W\ W\ '''$$

$$- cG\ (W\ 'W - W\ W\ ')]_0^L \qquad \text{(A2)}$$

The boundary conditions are received when the expression in the brackets in (A2) is equal to zero both for x = 0 and x = L. Observe first that $W_k(0) = W_k'(0) = 0$, $k = i,j$, these are the customary boundary conditions for a cantilever beam. Now rearrange the terms in the brackets in (A2) to arrive at

$$[(W_i''' - cG\, W_i')W_j - (W_j''' - cG\, W_j')W_i - W_i''W_j' + W_j''W_i']_0^L = 0$$

This expression is equal to zero when

$$W_k'''(L) - cG\, W_k'(L) = 0 \text{ and } W_k''(L) = 0 \quad (k = i,j) \qquad \text{(A3)}$$

The first of these conditions is not intuitively obvious. It requires that the resultant shear force at the free end become equal to zero. This is equivalent to saying that the shear force in the viscoelastic layer must be equal and opposite to the shear force in the elastic layer.

A complete set of bondary conditions is then

$$W = 0 \text{ and } W' = 0 \qquad \text{for } x = 0 \qquad \text{(A4)}$$

and

$$W''' - cG\, W' = 0 \text{ and } W'' = 0 \quad \text{for } x = L \qquad \text{(A5)}$$

These boundary conditions and the differential equation of motion are used to determine the system response.

APPENDIX B

BOUNDARY CONDITIONS AND STABILITY FOR THE PURELY VISCOUS CASE

To check if the analogous boundary conditions at x = L are also suitable for a purely viscous case, consider the equation

$$y'''' - cG\,\dot{y}'' + (\rho/EI)\,\ddot{y} = 0 \qquad \text{(B1)}$$

This equation differs from Eq. (16) in that the second term contains the time derivative of y''.

Consider the following energy equation

$$I = \int_0^t \int_0^L [y'''' - cG\,\dot{y}'' + (\rho/EI)\,\ddot{y}]\,\dot{y}\,dx\,dt = 0 \qquad \text{(B2)}$$

Integrating by parts and using the relation

$$\int_0^L \ddot{y}\,\dot{y}\,dx = (1/2)(dx/dt)\int_0^L (\dot{y})^2\,dx$$

leads to

$$1/2\int_0^L [(y'')^2 + (\rho/EI)(\dot{y})^2]dx + \int_0^t\int_0^L cG(\dot{y}')^2\,dx\,dt$$

$$+\int_0^t [(y'''\dot{y} - y''\dot{y}') - cG\,\dot{y}'\dot{y}\,]_0^L\,dt - \phi_o = 0 \qquad \text{(B3)}$$

where ϕ_o is the initial energy.

The third integral becomes

$$\int_0^t \left[(y''' - cG\,\dot{y}')\dot{y} - y''\dot{y}' \right]_{x=L} dt = 0. \qquad \text{(B4)}$$

Thus the boundary conditions for x = L are obtained when the expression in the brackets in (B3) is zero.

If again we assume the solution is separable as before, namely $y(x,t) = W(x)\exp(-wt)$, this leads to similar boundary conditions as those expressed by (A4) and (A5), namely

$$W''' - w\,cG\,W' = 0 \quad \text{and} \quad W'' = 0 \quad \text{for } x = L \qquad \text{(B5)}$$

The only difference being the appearence of w in the condition

$$W''' - w\,cG\,W' = 0\ .$$

In summary, we have used the energy method to derive a set of boundary conditions similar to those derived for the multilayer system with the exceptions noted above.

STABILITY OF THE SOLUTION

Notice that (B3) may be written, after taking advantage of the boundary conditions (B5), in the form :

$$1/2\int_0^L [(y'')^2 + (\rho/EI)(\dot{y})^2]\, dx = \phi_o - cG\int_0^t\int_0^L (\dot{y}')^2\, dx\, dt \qquad (B6)$$

Both integrands and ϕ_o are always positive or zero. It follows therefore, that for $t \longrightarrow 0$, we have $y \longrightarrow 0$, thus the solution to (B1) is stable.

APPENDIX C

PROOF THAT THE DERIVED SOLUTION IS SOLUTION OF THE BOUNDARY VALUE PROBLEM

Equation (30) is rewritten here

$$W(x) = W''(0)/(u^2 - v^2)\ [\cosh ux - \cosh vx - A\ (v \sinh ux - u \sinh vx)/uv] \qquad (C1)$$

where $A = (u^2 \cosh uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL)$

The first four derivatives with respect to x are

$$W'(x) = k\ [(u \sinh ux - v \sinh vx) - A\ (\cosh ux - \cosh ux)] \qquad (C2)$$

where $k = W''(0)\ /\ (u^2 - v^2)$

$$W''(x) = k\ [u^2 \cosh ux - v^2 \cosh vx - A\ (u \sinh ux - v \sinh vx)] \qquad (C3)$$

$$W'''(x) = k\ [u^3 \sinh ux - v^3 \sinh vx - A\ (u^2 \cosh ux - v^2 \cosh vx)] \qquad (C4)$$

$$W''''(x) = k\ [u^4 \cosh ux - v^4 \cosh vx - A\ (u^3 \sinh ux - v^3 \sinh vx)] \qquad (C5)$$

Making use of the derivatives given above, we will check that the boundary conditions (A4) and (A5), derived in Appendix A are satisfied.

(a) $W = 0$ at $x = 0$

Substitution of $x = 0$ in (C1) yields

$$W(0) = k\ [\cosh 0 - \cosh 0 - A\ (v \sinh 0 - u \sinh 0)/uv]$$

Noting that $\cosh 0 = 1$ and $\sinh 0 = 0$, the following is received

$$W(0) = k\ [1 - 1 - A\ (0 - 0)] = 0$$

Thus, the first boundary condition is satisfied.

(b) $W' = 0$ at $x = 0$

The same process as in (a) is repeated, and the result is

$$W'(0) = k\,[0 - 0 - A\,(1 - 1)] = 0$$

(c) $W'' = 0$ at $x = L$

Substitute $x = L$, and the value of A into (C3),simplfy an receive

$$W''(L) = k\,[u^2 \cosh uL - v^2 \cosh vL - u^2 \cosh uL + v^2 \cosh vL) = 0$$

(d) $W''' - cG\,W' = 0$ at $x = L$

To show that the boundary condition (d) is met by our solution, first recall from IV. that

$$cG = (u^2 + v^2)$$

and

$$A = (u^2 \cosh uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL)$$

These two values inserted into (d) yield after manipulation

$$W'''(L) - cG\,W'(L) = [k/(u \sinh uL - v \sinh vL)][T.E.]$$

where T.E. is the transcendental equation (28). Thus the fourth boundary condition is satisfied whenever the transcendental equation is satisfied.

The next step is to check that the complete solution also satisfies equation (17).

Assume as before

$$y(x,t) = W(x) \exp(-wt)$$

Observe the following relations

$$y''''(x,t) = W''''(x) \exp(-wt)$$

$$y''(x,t) = W''(x) \exp(-wt)$$

$$\ddot{y}(x,t) = (w)^2 W(x) \exp(-wt) = (w)^2 y(x,t)$$

Equation (17),after substitution of the above relations yields

$[W''''(x) - cG\, W''(x) + (w/wo)^2 W(x)] \exp(-wt) = 0$, and $\exp(-wt)$ is always positive for $0 < t < \infty$.

The expression in brackets has been shown to be the solution of the boundary value problem for the determination of W(x). Therefore y(x,t) is the solution of the given boundary value problem.

APPENDIX D

DETERMINATION OF THE COMPLEX FREQUENCIES u AND v

The complex frequencies u and v are determined by finding the poles of the transformed displacement function W[p]. The poles are received when the denominator of W[p] is equated to zero. This task is required in order to evaluate the inverse Laplace transform $L^{-1}\{\bar{W}[p]\}$.

Recall that the denominator of $\bar{W}[p]$ is given by

$$p^4 - rp^2 + s^2 = 0 = (p^2 - u^2)(p^2 - v^2) \tag{D1}$$

$$= p^4 - p^2(u^2 + v^2) + u^2 v^2$$

where $r = cG$; $s^2 = (w/w_0)^2$; $(w_0)^2 = EI/\rho$

Equate coefficients in (D1) and receive

$s^2 = (w/w_0)^2 = u^2 v^2$; $r = cG = u^2 + v^2$

A formal solution to (D1) gives

$$p_{1,2} = u_{1,2} = \pm \{1/2[r + (r^2 - 4s^2)^{1/2}]\}^{1/2} \tag{D2}$$

$$p_{3,4} = v_{1,2} = \pm \{1/2[r - (r^2 - 4s^2)^{1/2}]\}^{1/2} \tag{D3}$$

In the case of a purely elastic beam, the viscoelastic term $cG = r$ is equated to zero and the result is

$$u_{1,2} = \pm (\pm is)^{1/2} = \pm b$$

$$v_{1,2} = \pm (\pm is)^{1/2} = \pm ib$$

APPENDIX E

PROOF THAT UPON REMOVAL OF THE VISCOELASTIC TERM cG, THE SOLUTION REDUCES TO THAT OF AN ELASTIC BEAM

The equation of motion for the multilayer beam is

$$y(x,t) = W(x)\ \exp(-wt)$$

$$W(x) = k\ [\cosh ux - \cosh vx - A\ (v \sinh ux - u \sinh vx)/uv]$$

where

$$k = W''(0)/(u^2 - v^2)$$

and

$$A = (u^2 \cosh uL - v^2 \cosh vL)/(u \sinh uL - v \sinh vL)$$

Upon insertion of the values of u and v for the elastic beam

$$k_1 = W''(0)/2b^2$$

Combination u_1/v_1 ($u_1 = b$, $v_1 = ib$)

$$W(x) = k_1\{\cosh bx - \cosh ibx - [b \cosh bL - ib \cosh ibL]\ x$$
$$(ib \sinh bx - b \sinh ibx)/(b \sinh bL - ib \sinh ibL)\ ib\ b\ )\}$$

Using $\sinh ibx = i \sin bx$ and $\cosh ibx = \cos bx$, we obtain

$$W(x) = k_1\{\cosh bx - \cos bx - [(\cosh bL + \cos bL)/(\sinh bL + \sin bL)]$$
$$x\ (\sinh bx - \sin bx)]\}\qquad (E1)$$

In the following pages we will show that Eq. (E1) is a solution of the purely elastic case as given in [2].

Combination u_1/v_2 ($u_1 = b$, $v_2 = -ib$)

Making use of the properties cosh $-ibx$ = cosh ibx = cos bx and sinh $-ibx$ = - sinh ibx = - i sin bx , equation (E1) is readily received.

The other two combinations ($u_2 = -b$, $v_1 = ib$ and $u_2 = -b$, $v_2 = -ib$) when inserted into $W(x)$ yield the same result given in (E1).

We will now check that (E1) is a solution to the elastic beam as given in [2].

The differential equation is

$$W'''' - b^4 W = 0, \text{ where } W = W(x), \quad 0 < x < L$$

The boundary conditions are

$$W = 0 \text{ and } W' = 0 \text{ at } x = 0$$

$$W'' = 0 \text{ and } W''' = 0 \text{ at } x = L$$

Let us compute the required derivatives.

$$W'(x) = k_2 b\ [s(L)\ (\sinh bx + \sin bx) - c(L)\ (\cosh bx - b \cos bx)]$$

where

$s(L) = \sin bL + \sinh bL$

$c(L) = \cosh bL + \cos bL$

$k_2 = k_1/s(L) = W''(0)/2b^2 (\sin bL + \sinh bL)$

$$W''(x) = k_2 b^2 [s(L)\ (\cosh bx + \cos bx) - c(L)\ (\sinh bx + \sin bx)]$$

$$W'''(x) = k_2 b^3 [s(L)\ (\sinh bx - \sin bx) - c(L)\ (\cosh bx + \cos bx)]$$

$$W''''(x) = k_2 b^4 [s(L)\ (\cosh bx - \cos bx) - c(L)\ (\sinh bx - \sin bx)]$$

$$W''''(x) = b^4 W(x)$$

From the above equation we see that the differential equation for the purely elastic case is satisfied.

We will now check that the solution satisfies the four boundary conditions.

1. $W = 0$ at $x = 0$

$$W(0) = k_2 [s(L)\ (1 - 1) - c(L)\ (0 - 0)] = 0$$

2. $W' = 0$ at $x = 0$

$$W'(0) = k_2 b\ [s(L)\ (0 + 0) - c(L)\ (1 - 1)] = 0$$

3. $W''L = 0$ at $x = L$

$$W''(L) = k_2 b^2 [s(L)\ (\cos bL + \cosh bL) - c(L)\ (\sinh bL + \sin bL)]$$

$$W''(L) = k_2 b^2 [s(L)\ c(L) - c(L)\ s(L)] = 0$$

4. $W''' = 0$ at $x = L$

$$W'''(L) = k_2 b^3 [s(L)\ (\sinh bL - \sin bL) - c(L)\ (\cosh bL + \cos bL)]$$

Insertion of $s(L) = \sinh bL + \sin bL$ and $c(L) = \cosh bL + \cos bL$ into $W'''(L)$, yields after manipulation

$$W'''(L) = k_2 b^3 [-(\cosh^2 bL - \sinh^2 bL) - (\sin^2 bL + \cos^2 bL) - 2 \cosh bL \cos bL]$$

Observe that $\cosh^2 bL - \sinh^2 bL = 1$ and $\sin^2 bL + \cos^2 bL = 1$

These two identities when inserted in $W'''(L)$ lead to

$$W'''(L) = k_2 b^3 [-1 - 1 - 2 \cosh bL \cos bL]$$
$$= - 2k_2 b^3 [1 + \cosh bL \cos bL] = 0$$

The above equation is identically zero by virtue of the fact that the quantity in brackets is the characteristic equation for the purely elastic cantilever beam [2].

In conclusion, we showed that upon removal of the viscoelastic term cG from the multilayer system roots u and v the following is observed:

1. The multilayer beam response y(x,t) reduces to that of a purely elastic beam.

2. The characteristic equation of the multilayer beam reduces to that of a purely elastic beam.

3. The boundary conditions of the multilayer beam reduce to those of the purely elastic beam.

4. The multilayer beam response y(x,t) satisfies all of the boundary conditions of the purely elastic beam upon setting the viscoelastic term cG equal to zero.

APPENDIX F

F.1 Comments about the computer program

The program is written to accomplish two tasks. The first task is to evaluate the complex frequencies u and v. The second task is to use the values of the complex frequencies to plot the response of the system as a function of both position and time.

The program is written in Turbo-Pascal. The first part of the program defines the constants and the variables. The inversion count is used to account for the various combinations of the roots u , v which appear in both the characteristic equation and the final solution y(x,t). The following part of the program defines the hyperbolic sine, cosine and the exponential functions. After making these definitions, a plotting routine is written with scaling so that the plots would fit on the monitor screen. The evaluation of the complex frequencies is carried out in the program using the algorithm given in the computational example. The program is based on the assumption that the imaginary part of the shear modulus is so small in comparison to its real part that it is ignored.

The program is designed to plot the response of the compound beam at the position x = 10 inches. The chosen frequency range is compatible with the restrictions and concerns reported in the discussion. The thickness of the viscoelastic material is varied from 0.10 to 5.0 inches while maintaining the thickness of the constraining layer at 0.01 inches. The central layer is chosen to have a 0.10 x 1.0 section. Note that the assumption of the constraining layers being thin in comparison to the central layer is maintained.

The second program is basically identical to the first program, with the exception of the computational algorithm used for the evaluation of $u_{1,2}/v_{1,2}$. To check the effect of the constraining and viscoelastic layers' thicknesses on the response of the system, both h_2 and h_3 are set as input variables to be supplied the program before every run. The frequency range and increment are set as variables to be input in a similar way as h_2 and h_3 .

The thickness of the central layer is arbitrarily set equal to 0.10 inches, and to maintain the assumption that the constraining layers are thin in comparison to the central layer, they are chosen to be 0.01 inches or $h_1/h_3 = 10$.

F.2.A FIRST PROGRAM LISTING

```
program analysis (input, output, uv_file);
const
   L = 10.0;
   h2 = 5.0;
   h2_doubled = 10.0;
var
   C,
   B,
   h3,
   h3_doubled,
   r_over_2,
   omega,
   omega0,
   sq_root_r_over_2,
   cos_theta,
   cos_alpha,
   u,
   v,
   equation_result,
   x,
   A,
   t,
   u_squared,
   v_squared,
   uL,
   vL,
   ux,
   vx,
   uv,
   cosh_uL,
   cosh_vL,
   sinh_uL,
   sinh_vL,
   exp_uL,
   exp_vL:
      real;
   inversion_count:
      integer;
   uv_file:
      text;

function sinh (x: real): real;
begin
   sinh := 0.5 * (exp (x) - 1 / exp (x))
end;

function cosh (x: real): real;
begin
  cosh := 0.5 * (exp (x) + 1 / exp (x))
end;

function pow (base, exponent: real): real;
begin
   pow := exp (ln (base) * exponent)
end;

procedure draw_axes (x, omega0, h2, h3: real);
var
   i: 1..4;
begin
   ClrScr;
```

```
    HiRes;
    HiResColor (7);
    writeln (' Graph for x = ', x:4:1, '; omega0 =', omega0, ';');
    writeln ('                        h2 =', h2:5:2, '; h3 =', h3:5:2);
    writeln (' 4in');
    for i := 1 to 4 do writeln;
    writeln (' 3in');
    for i := 1 to 4 do writeln;
    writeln (' 2in');
    for i := 1 to 4 do writeln;
    writeln (' 1in');
    for i := 1 to 4 do writeln;
    writeln (' 0in');
    write (
       0s          1s          2s          3s          4s          5s          6s'
          );
    Draw (50, 20, 50, 180, 7);
    Draw (50, 20, 55, 20, 7);
    Draw (50, 60, 55, 60, 7);
    Draw (50, 140, 55, 140, 7);
    Draw (50, 180, 55, 180, 7);
    Draw (50, 180, 600, 180, 7);
    for i := 1 to 6 do
       Draw (50+90*i, 178, 50+90*i, 182, 7)
end;

procedure plot_point (value, t: real);
var
   x, y: integer;
begin
   if (abs (value) <= 4.0) then
   begin
      y := round (value / 4.0 * 160);
      y := 180 + -1 * y;
      x := round (t / 6.0 * 540);
      x := 50 + x;
      Plot (x, y, 7)
   end
end;

begin
   assign (uv_file, 'uv_file.out');
   rewrite (uv_file);
   write ('h3? ');
   readln (h3);
   write ('x? ');
   readln (x);
   h3_doubled := 2 * h3;
   omega0 := 5.849775E4 *
             sqrt ((1 + pow (h3_doubled, 3)) / (1 + h3_doubled));
   draw_axes (x, omega0, h2, h3);
   omega := 1.0;
   while (omega < 10.0) do
   begin
   C := (1 + (1.05 / h2_doubled)) * (1 / 2.500625E6);
   B := 800.0 * pow (omega / 20.0, 0.2341);
   r_over_2 := C * B / 2;
   sq_root_r_over_2 := sqrt (r_over_2);
   cos_theta := cos (arctan (2 * omega / (omega0 * C * B)));
   cos_alpha := cos (arctan (1 / sqrt (cos_theta)));
   u := sq_root_r_over_2 * sqrt (1 + 1 / cos_theta);
   v := sq_root_r_over_2 *      (1 / cos_alpha);
      u_squared := sqr (u);
      v_squared := sqr (v);
      uL := u * L;
      VL := V * L;
```

```
      uv := u * v;
      exp_uL := exp (uL);
      exp_vL := exp (vL);
      cosh_uL := cosh (uL);
      cosh_vL := cosh (vL);
      sinh_uL := sinh (uL);
      sinh_vL := sinh (vL);
      equation_result :=
          (sqr (u_squared) + sqr (v_squared)) *
          (cosh_uL * cosh_vL) - uv * (u_squared + v_squared) *
          (sinh_uL * sinh_vL) - (2 * u_squared * v_squared);
      if (equation_result < 0.000001) then
      begin
          writeln (uv_file);
          writeln (uv_file, 'omega: ', omega);
          writeln (uv_file, 'h2: ', h2);
          writeln (uv_file, 'omega0: ', omega0);
          writeln (uv_file, '(u,v) = (', u, ',', v, ')');
          A := (u_squared * cosh_uL - v_squared * cosh_vL) /
                  (u * sinh_uL - v * sinh_vL);
          ux := u * x;
          vx := v * x;
          t := 0.0;
          while (t <= 6.0) do
          begin
              equation_result :=
                  exp (-1 * omega * t) / (u_squared - v_squared) *
                  (cosh (ux) - cosh (vx) -
                  A * (v * sinh (ux) - u * sinh (vx)) / uv);
              plot_point (equation_result, t);
              t := t + 0.001
          end
      end;
   omega := omega + 1.0
   end;
   close (uv_file)
end.
```

F.2.B. SECOND PROGRAM LISTING

```
program analysis (input, output, uv_file);
const
   L = 10.0;
   x = 10.0;
   tolerance = 0.001;
   timing_resolution = 0.0001;
var
   C,
   G,
   h2,
   h3,
   r,
   r_over_2,
   radical,
   omega,
   omega_low,
   omega_high,
   omega_increment,
   omega0,
   sq_root_r_over_2,
   u,
   v,
   equation_result,
   A,
   t,
   u_squared,
   v_squared,
   uL,
   vL,
   ux,
   vx,
   uv,
   cosh_uL,
   cosh_vL,
   sinh_uL,
   sinh_vL,
   exp_uL,
   exp_vL:
      real;
   inversion_count:
      integer;
   uv_file:
      text;

function sinh (x: real): real;
begin
   sinh := 0.5 * (exp (x) - 1 / exp (x))
end;

function cosh (x: real): real;
begin
  cosh := 0.5 * (exp (x) + 1 / exp (x))
end;

function pow (base, exponent: real): real;
begin
   pow := exp (ln (base) * exponent)
end;

procedure draw_axes (x, omega0, omega_low. omega_high, h2, h3: real);
var
   i: 1..4;
begin
```

```
   ClrScr;
   HiRes;
   HiResColor (7);
   writeln (' Graph for x = ', x:4:1, '; omega0 =', omega0, ';');
   write   ('                h2 =', h2:6:2, '; h3 =', h3:6:2, '; omega_range = ');
   writeln (omega_low:4:1, '...', omega_high:4:1);
   writeln (' 4in');
   for i := 1 to 4 do writeln;
   writeln (' 3in');
   for i := 1 to 4 do writeln;
   writeln (' 2in');
   for i := 1 to 4 do writeln;
   writeln (' 1in');
   for i := 1 to 4 do writeln;
   writeln (' 0in');
   write (
      0s        0.5s          1s         1.5s         2s         2.5s          3s'
      );
   Draw (50, 20, 50, 180, 7);
   Draw (50, 20, 55, 20, 7);
   Draw (50, 60, 55, 60, 7);
   Draw (50, 140, 55, 140, 7);
   Draw (50, 160, 55, 180, 7);
   Draw (50, 180, 600, 180, 7);
   for i := 1 to 6 do
      Draw (50+90*i, 178, 50+90*i, 182, 7)
end;

Procedure plot_point (value, t: real);
Var
   x, y: integer;
begin
   if (abs (value) <= 4.0) then
   begin
      y := round (value / 4.0 * 160);
      y := 180 + -1 * y;
      x := round (t / 3.0 * 540);
      x := 50 + x;
      Plot (x, y, 7)
   end
end;

begin
   assign (uv_file, 'uv_file.out');
   rewrite (uv_file);
   write ('h2? ');
   readln (h2);
   write ('h3? ');
   readln (h3);
   write ('omega low, omega high, omega increment? ');
   readln (omega_low, omega_high, omega_increment);
   omega0 := 5.849775E4 * sqrt ((0.001 + 2 * pow (h3, 3)) / (0.1 + 2 * h3));
   draw_axes (x, omega0, omega_low, omega_high, h2, h3);
   omega := omega_low;
   while (omega <= omega_high) do
   begin
      C := (0.01 + 0.2 * h2 + 0.1 * h3) /
           (5e6 * h2 * (0.001 + 2 * pow (h3, 3)));
      B := 800.0 * pow (omega / 20.0, 0.2341);
      r := C * B;
      r_over_2 := r / 2;
      radical := sqr (r) - 4 * sqr (omega / omega0);
      if (radical < 0) then
         radical := 0
      else
         radical := sqrt (radical) / 2;
```

```
        u := sqrt (r_over_2 + radical);
        v := sqrt (r_over_2 - radical);
        u_squared := sqr (u);
        v_squared := sqr (v);
        uL := u * L;
        vL := v * L;
        uv := u * v;
        exp_uL := exp (uL);
        exp_vL := exp (vL);
        cosh_uL := cosh (uL);
        cosh_vL := cosh (vL);
        sinh_uL := sinh (uL);
        sinh_vL := sinh (vL);
        equation_result :=
            (sqr (u_squared) + sqr (v_squared)) *
            (cosh_uL * cosh_vL) - uv * (u_squared + v_squared) *
            (sinh_uL * sinh_vL) - (2 * u_squared * v_squared);
        if (equation_result < tolerance) then
        begin
            writeln (uv_file);
            writeln (uv_file, 'omega: ', omega);
            writeln (uv_file, 'h2: ', h2);
            writeln (uv_file, 'omega0: ', omega0);
            writeln (uv_file, '(u,v) = (', u, ',', v, ')');
            A := (u_squared * cosh_uL - v_squared * cosh_vL) /
                  (u * sinh_uL - v * sinh_vL);
            ux := u * x;
            vx := v * x;
            t := 0.0;
            while (t <= 3.0) do
            begin
                equation_result :=
                    exp (-1 * omega * t) / (u_squared - v_squared) *
                    (cosh (ux) - cosh (vx) -
                    A * (v * sinh (ux) - u * sinh (vx)) / uv);
                plot_point (equation_result, t);
                t := t + timing_resolution
            end
        end;
    omega := omega + omega_increment
    end;
    close (uv_file)
end.
```